Slices in Equivariant Topology: A Short Introduction

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Abstract

We introduce, with many examples, the slice theorem for (nice) G-spaces.

Let $p : P \to B$ be a principal G-bundle. This means that it is a fiber bundle equipped with a free action of G on the total space P, restricting to a transitive action on each fiber. In other words, for each $b \in B$, the fiber $P_b := p^{-1}(b)$ is a G-*torsor*: a space which, for any choice of point $x \in P_b$, is G-equivariantly homeomorphic to G via a unique map sending x to the identity of G. Moreover, this structure yields an isomorphism $B \cong P/G$ under P. To say the slogan:

> *A G-torsor is a copy of G which has forgotten its identity; a principal bundle is a continuous family of G-torsors parametrized by P/G.*

Now let X be a G-space. The quotient map $p : X \to X/G$ is not necessarily a principal G-bundle, since the action of G on X may not be free. But for each point $b \in X/G$, while the fiber $p^{-1}(b)$ may not have a free action, it is indeed a transitive G-space. A choice of any point $x \in p^{-1}(b)$—witnessing $p^{-1}(b)$ as the orbit $G \cdot x$—induces a G-equivariant homeomorphism between the fiber X_b and the coset space G/H, where $H := G_x$ is the stabilizer of x, sending x to the distinguished coset eH. More succinctly:

> *A G-orbit is a copy of G/H which has forgotten its distinguished point; a G-space is a continuous family of G-orbits parametrized by X/G.*

In light of this analogy, it is natural to ask whether the local trivializations of a principal bundle have an analogue for G-spaces. In general, the action of G on X may be sufficiently complicated that no such analogue can be found. However, in the case where G is compact and acts smoothly on a manifold, the *slice theorem* provides exactly such an analogue.

We may think of a trivialization above U of a principal G-bundle $p : P \to B$ as a continuous choice of identity for each fiber above U. Indeed, the data of a local trivialization $G \times U \xrightarrow{\cong} p^{-1}(U)$ is equivalent to the data of a section $s : U \hookrightarrow P$. By transitivity of the G-action on fibers, $p^{-1}(U)$ is just the orbit $G \cdot s(U)$; from this perspective, the trivialization is the map $(g, b) \mapsto g \cdot s(b)$.

Now consider this perspective in the context of a general G-space X. Since G-orbits are supposed to be G/Hs which have forgotten their distinguished points, a natural first try is to say that a trivialization above U is continuous choice of distinguished point in each orbit, i.e. a section $s : U \hookrightarrow X$. Since G acts transitively on orbits, we again have an equality $p^{-1}(U) = G \cdot s(U)$. The candidate trivialization $(g, G \cdot x) \mapsto g \cdot s(G \cdot x)$ is always surjective, by transitivity of the G-action on orbits. However, it fails to be injective when the action is not free.

Example 1. Consider the real sign representation of C_2: the nontrivial element $\tau \in C_2$ acts on $\mathbb{R}$ by the reflection $x \mapsto -x$. In this case we even have a global section of the quotient, given by $C_2 \cdot x \mapsto |x|$. However, the candidate map sends both $(e, C_2 \cdot 0)$ and $(\tau, C_2 \cdot 0)$ to 0.

More generally, if $x \in s(U)$ is stabilized by $h \in G_x$, then the candidate map sends both $(e, G \cdot x)$ and $(h, G \cdot x)$ to x. This observation suggests taking some kind of quotient of $G \times U$ by a stabilizer. However, it is not immediately clear how to do this: first, the points in $s(U)$ may have many different stabilizers; and second, there is no action of any nontrivial subgroup of G on either U or $s(U)$ by which to quotient.

Return for a moment to the free case, i.e. that of a principal bundle $p : P \to B$. For any point $x \in P$, its orbit is identical with the fiber $P_{p(x)}$. The data of a neighborhood of x of the form $p^{-1}(U)$, for some neighborhood U of $p(x)$, is therefore the same as the data of a neighborhood of the orbit x which is invariant under the action of G. To exhibit a local trivialization of P is thus to give, for each $x \in P$, a G-invariant neighborhood $\mathcal{N}$ of x together with a G-homeomorphism of $\mathcal{N}$ to the product $G \times p(\mathcal{N})$. Moreover, for any such $\mathcal{N}$, we can find a set $S \subseteq \mathcal{N}$ isomorphic to $p(\mathcal{N})$ which corresponds across this homeomorphism to the "slice" $\{e\} \times p(\mathcal{N})$. From the perspective of a section $s : U \hookrightarrow P$, S is the set $s(U)$ and $\mathcal{N}$ is $G \cdot s(U)$. Observe that S meets each fiber exactly once.

The advantage of focusing our attention on a point $x \in X$ of the total space is that x comes equipped with its stabilizer $H := G_x$. This resolves the first issue above. For the second, rather than asking that the slice S meet each orbit exactly once—as is the case for the image of a section—we can instead ask that S be H-invariant. In this case, we can replace the product $G \times U \cong G \times s(U)$ with the balanced product[1] $G \times_H S$. This balanced product exactly reflects the fact that the process of forming the quotient X/G is not uniform: the bigger the stabilizer of a point, the fewer the points in its orbit. Indeed, for a point $y \in S$ stabilized by $h \in H$, we will have an identification (g, y) with $(gh, h^{-1}y) = (gh, y)$, so that y will have fewer associated points in the balanced product.

Definition 2. Let X be a G-space and $x \in X$ with stabilizer $H := G_x$. A *slice at x* is an H-invariant subspace $x \in S \subseteq X$ so that:

1. the orbit $G \cdot S$ is open in X;
2. the natural map

$$\begin{aligned} \phi_S \colon G \times_H S &\to G \cdot S \\ (g, s) &\mapsto g \cdot s \end{aligned}$$

   is a G-homeomorphism.

Since the quotient $p : X \to X/G$ is open, $G \cdot S$ is open in X if and only if its image $p(G \cdot S) = p(S)$ is open in X/G. Thus, the first condition translates the requirement that trivializations of principal bundles be defined on open sets of the base.

Convention 3. When $X = M$ is a smooth manifold with smooth G-action, we say that S is a *smooth slice* if S is a smooth submanifold of M. When it is clear that we are working in the smooth setting, we will more simply say "slice" to mean "smooth slice."

The map ϕ_S is always continuous, equivariant, surjective, and open; in the smooth setting, ϕ_S is always a smooth map. (Openness is the hardest to see: any topological action map $a : G \times S \to G \cdot S$ is always open, and $\phi_S \circ q = a$, where $q : G \times S \to G \times_H S$ is the quotient.) The only condition to check is therefore injectivity. Before looking at some examples, it will be helpful to have the following alternative characterization of this property.

Proposition 4. *Let X be a G-space and $S \subseteq X$ be H-invariant for some $H \leq G$. Then the natural map ϕ_S is injective if and only if S satisfies the* slice condition*: whenever $g \cdot S \cap S \neq \varnothing$, we have $g \in H$.*

Proof. If ϕ_S is injective and $g \cdot S \cap S \neq \varnothing$, then there are $s_1, s_2 \in S$ with $g \cdot s_1 = s_2$, i.e. $\phi_S(g, s_1) = \phi_S(e, s_2)$. By injectivity, $(g, s_1) \sim (e, s_2)$, i.e. there is $h \in H$ with $(g, s_1) = (eh, h^{-1}s_2)$, so in particular $g = h$. Conversely, if S satisfies the slice condition and

$$g_1 \cdot s_1 = \phi_S(g_1, s_1) = \phi_S(g_2, s_2) = g_2 \cdot s_2,$$

then $g_2^{-1} g_1 \cdot s_1 = s_2$, so by the slice condition $g_2^{-1} g_1 \in H$. But this implies exactly that $(g_1, s_1) = (g_2, s_2)$ are equal in $G \times_H S$. ◻

Example 5. Let $p : P \to B$ be a principal G-bundle and let $b \in B$. Let U be a trivializing neighborhood of B with trivialization $\phi : G \times U \xrightarrow{\cong} p^{-1}(U)$. By the previous discussion, the data of ϕ is equivalent to that of a section $s : U \hookrightarrow P$. Then $s(U)$ is a slice at $s(b)$. Indeed, the stabilizer $G_{s(b)}$ is trivial; since $G \times_e s(U) \cong G \times s(U) \cong G \times U$, the map $\phi_{s(U)}$ recovers the trivialization ϕ.

Conversely, let S be a slice at x. Since $p(S) = p(G \cdot S)$ is open in B, we have a candidate trivialization

$$\begin{aligned} G \times p(S) &\to G \cdot S \\ (g, p(x)) &\mapsto g \cdot x. \end{aligned}$$

[1] Let H act on X on the right and Y on the left; then the *balanced product* $X \times_H Y$ is the quotient of $X \times Y$ by the relation identifying (x, y) with $(xh, h^{-1}y)$ for each $h \in H$. In the case where $X = G$ is a group with H a subgroup acting on G via the right regular action, this construction is also called *induction*; it is the left adjoint to the restriction functor from G-spaces to H-spaces. In particular, $G \times_H S$ carries a natural G-action via the left regular action on the G factor.

We need only to show that the continuous surjection $S \to p(S)$ is a homeomorphism. Since $(G \times_e S)/G \cong S$ and ϕ_S is equivariant, we may realize this map as $\phi_S/G : (G \times S)/G \to (G \cdot S)/G$. Since ϕ_S is a G-isomorphism, so is ϕ_S/G. Therefore the slice determines a local trivialization of the principal bundle.

In all, for each $b \in B$ this discussion establishes a canonical bijection

$$\{\text{slices at some } x \in P_b\} \Longleftrightarrow \{\text{neighborhoods } U \text{ of } b \text{ equipped with trivializations } G \times U \xrightarrow{\cong} p^{-1}(U)\}.$$

Example 6. Consider again the sign representation of C_2 on $\mathbb{R}$. Let $x = 0$, so that $H = C_2$. Then any neighborhood $S := (-\epsilon, \epsilon)$ of 0 is a slice; we have $C_2 \times_{C_2} S \cong S$ and the map ϕ_S is just the identity map from S to itself. Note that $\{0\}$ itself is not a slice, since $C_2 \cdot \{0\} = \{0\}$ is not open in $\mathbb{R}$.

More generally, for any G-space X and any G-fixed $x \in X^G$, any open G-invariant neighborhood S of x (for instance, X itself) is a slice at x. We have $G \times_G S \cong S$, $G \cdot S = S$, and the map ϕ_S is again just the identity on S.

Example 7. Again considering the sign representation, let $x \neq 0$, so that $H = \{e\}$. Then any open interval S around x which does not contain 0 is a slice. Indeed, $C_2 \cdot S = S \sqcup -S$; ϕ_S identifies $\{e\} \times S$ with S and $\{\tau\} \times S$ with $-S$.

If S is a neighborhood of x containing 0, then this is false. The issue is that $C_2 \cdot S$ contains only one copy of 0, whereas $C_2 \times_e S$ contains two distinct copies, $(e, 0)$ and $(\tau, 0)$. Similarly, if S is a neighborhood of x containing both y and $-y$ for some $y \neq 0$, then ϕ_S fails to be injective, since both (e, y) and $(\tau, -y)$ map to y.

More generally, for any G-space X and any $x \in X$ with trivial stabilizer, an open neighborhood S of x is a slice if and only if each point in S has trivial stabilizer and the G-translates of S are disjoint.

Example 8. Consider the action of $O(n)$ on $\mathbb{R}^n$. The stabilizer of a point $x \neq 0$ is the copy of $O(n-1)$ which fixes the radial line through x. Take a small interval $S := \{x + tx : t \in (-\epsilon, \epsilon)\}$ around x on this line. Then S is H-fixed, and $O(n) \cdot S$ is an open thickening of the sphere of radius $||x||$. Moreover, since $O(n-1)$ fixes S pointwise, $O(n) \times_{O(n-1)} S$ is just $O(n)/O(n-1) \times S$, which is indeed G-isomorphic to $O(n) \cdot S$.

Example 9. Let the complex unit circle S^1 act on $\mathbb{C}^2$ with weights $(1, 2)$, i.e. via the action $w \cdot (z_1, z_2) := (wz_1, w^2 z_2)$. Let $x := (0, 1)$; the stabilizer H of x is the group $\{\pm 1\}$. We form a slice by fixing the phase in the second coordinate, so let $S := \{(z, 1+t) : t \in (-\epsilon, \epsilon)\}$. Then H fixes the second coordinate of S, so acts on S via $-1 \cdot (z, 1+t) = (-z, 1+t)$. The quotient $S^1 \times_H S$ identifies triples of the form $(w, z, 1+t)$ and $(-w, -z, 1+t)$; the map ϕ_S sends $(w, z, 1+t)$ to $(wz, w^2(1+t))$.

We need only to show injectivity of ϕ_S. Suppose $(w, z, 1+t)$ and $(w', z', 1+t')$ are such that $w^2(1+t) = w'^2(1+t')$ and $wz = w'z'$. Since $|w| = 1$, the first equality implies $t = t'$, and so $w = \pm w'$. Now $z = \pm z'$ with the same parity, which exactly establishes that these two triples are equal mod H.

Remark 10. We record here an abstract characterization of the map ϕ_S from Definition 2. The subgroup inclusion $H \leq G$ induces an adjoint pair $\mathrm{ind}_H^G : \mathrm{Top}^H \rightleftarrows \mathrm{Top}^G : \mathrm{res}_H^G$, where res_H^G is the restriction functor and the induction functor ind_H^G is given by the balanced product $G \times_H (-)$. Since $S \subseteq X$ is H-invariant, there is an inclusion of H-spaces $\iota : S \hookrightarrow \mathrm{res}_H^G X$. The map ϕ_S is the transpose of the inclusion under the induction-restriction adjunction.

We now turn to the question of existence of slices. Motivated by the analogy to principal bundles, we may hope that slices always exist. As the following two examples show, this is not the case in general.

Consider first any free, non-transitive Hausdorff G-space X with dense orbits, such as the action of $\mathbb{Z}$ on the circle S^1 by an irrational rotation. Because orbits are dense, any nonempty G-invariant open set must be all of X (note that its complement is a G-invariant closed set); so if S is a slice, we must have $G \cdot S = X$. Because the action is free, the map ϕ_S is a G-homeomorphism $G \times S \cong G \times_e S \xrightarrow{\cong} G \cdot S = X$. Passing to orbits, this induces a homeomorphism $S \cong X/G$. Since S is a subspace of a Hausdorff space, hence Hausdorff, X/G is Hausdorff. But X/G is not even T_1—a set in the quotient topology is closed iff its preimage is closed, so X/G is T_1 iff each orbit in X is closed, but a dense closed set is all of X. Therefore the action had to be transitive, a contradiction. The issue here is a lack of compactness of G (or, more generally, of *properness* of the action); this is what allows it to act with dense orbits.

Consider now $X := \{a, b\} \sqcup \{x, y\}$, where both components are given the indiscrete topology. Let C_2 act on X by swapping a with b and x with y. Suppose S is a slice at a. Since S must have disjoint G-translates, we cannot have $b \in S$, so the slice is either $\{a\}$, $\{a, x\}$, or $\{a, y\}$. In any of these cases, $G \times S$ is a product of discrete spaces, hence discrete; while $G \cdot S$ is not discrete. The issue in this case is a lack of regularity of X.

These two counterexamples illustrate the need for some compactness and regularity hypotheses. Fortunately, with relatively mild such hypotheses, slices always exist. (In fact, conditions can be weakened considerably; see [Pal61] for details.)

Theorem 11 (Slice Theorem [Mos57]). *Suppose G is a compact Lie group acting on a completely regular*[2] *Hausdorff space X. Then for any $x \in X$, there exists a slice at x.*

While a complete proof is outside our scope—the author quite likes the exposition of [Bre72]—we sketch the considerably simpler proof in the case where X is a smooth manifold with smooth G-action. First, observe that we may find a G-invariant Riemannian metric on M by starting with any Riemannian metric and using compactness of G to average it over orbits. Next, take a tubular neighborhood of the orbit $G \cdot x$ in M; G-invariance of the metric implies that this neighborhood is G-invariant. For sufficiently small radius, the neighborhood is G-diffeomorphic to the normal bundle of the orbit; we may then take the slice to be the fiber of this normal bundle over x.

References

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[Pal61] Richard S. Palais. "On the Existence of Slices for Actions of Non-Compact Lie Groups". In: *Annals of Mathematics* 73.2 (1961), pp. 295–323. URL: http://www.jstor.org/stable/1970335.

[2] A space is *completely regular* if points can be separated from closed sets by maps to the interval; completely regular Hausdorff spaces are also called *Tychonoff* spaces. The latter condition is equivalent to the existence of an embedding of the space into a compact Hausdorff space.