Oops All Algebra: Homework IV

Mathcamp 2025

Problem 1 (recommended)**.** Complete the proof that equivalence defines an equivalence relation on the set of morphisms from $x \to y$ in a quasicategory C.

Problem 2 (recommended)**.** Verify that the one-truncation $\tau_1 C$ of a quasicategory C is a category. In particular, convince yourself that composition is well-defined and associative.

Problem 3 (optional)**.** Find a quasicategory C which is not isomorphic to the nerve of a category.

Problem 4 (optional)**.** Convince yourself that there is a bijection

$$\textsc{Cat}_1(\tau_1 C, \mathcal{D}) \cong \textsc{Cat}(C, N\mathcal{D})$$

for any quasicategory C and category $\mathcal{D}$.

Problem 5 (optional)**.** Given a simplicial set X and zero-cells x and y, let $\mathrm{Map}_X(x, y)$ be the simplicial set with

$$\mathrm{Map}_X(x, y)_n = \{f : \Delta^{n+1} \to X : f(0) = x, f(n+1) = y\}.$$

(a) Define the face and degeneracy maps on $\mathrm{Map}_X(x, y)$, and convince yourself that they satisfy the simplicial identities.

(b) Describe $\mathrm{Map}_X(x, y)_n$ more explicitly in the case where $X = NC$ is the nerve of a category.

(c) Convince yourself that a morphism $f : y \to z$ in a quasicategory C induces a simplicial map $f_* : \mathrm{Map}_C(x, y) \to \mathrm{Map}_C(x, z)$. Convince yourself that a morphism $g : z \to x$ induces a simplicial map $g^* : \mathrm{Map}_C(x, y) \to \mathrm{Map}_C(z, y)$.

(d) Convince yourself that, if C is a quasicategory, then $\mathrm{Map}_C(x, y)$ is a quasicategory. (In fact it is a quasigroupoid, but this proof is outside our scope.)