Oops All Algebra: Homework III

Mathcamp 2025

Problem 1 (recommended)**.** Convince yourself that any functor $F : \Delta^{\mathrm{op}} \to \textsc{Set}$ satisfies the simplicial identities, and conversely that any simplicial set determines such a functor, i.e. that these are equivalent definitions.

Problem 2 (recommended)**.** Let X be a simplicial set with unique fillers of all inner horns. In class, we defined the 1-*truncation* $\tau_1 X$ of X as the category whose objects are the 0-simplices X_0 and whose morphisms are the 1-simplices, with sources and targets given by the face maps, identities by the degeneracies, and composition by the inner horn fillers. Show that $X \cong N\tau_1 X$. (Hint: once you have defined maps $f_n : X_n \to (N\tau_1 X)_n$ for each n, show that they are bijections via induction on n.)

Problem 3 (recommended)**.** Convince yourself that a category C is a groupoid, i.e. that all its morphisms are isomorphisms, if and only if NC has unique fillers of all horns.

Problem 4 (optional, Homotopy Hypothesis II)**.** On Homework II, we defined the *singular simplicial set* and *geometric realization*, functors

$$\textsc{Top} \underset{|\text{-}|}{\overset{\mathrm{Sing}}{\rightleftarrows}} \mathrm{s}\textsc{Set}.$$

(a) Show that if X is a simplicial set and Y is a topological space, then there is a bijection

$$\textsc{Top}_1(|X|, Y) \cong \mathrm{s}\textsc{Set}_1(X, \mathrm{Sing}\, Y).$$

(If you know what a natural transformation is, show that this bijection is natural in X and Y, i.e. that there is an adjunction $|-| \dashv \mathrm{Sing}$.)

(b) A *Kan complex* is a simplicial set K with (not necessarily unique) fillers of all horns, i.e. so that any map $\Lambda^n_j \to K$ admits a lift

$$\begin{tikzcd} \Delta^n \arrow[r, dashed] & K. \\ \Lambda^n_j \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Show that the singular simplicial complex of any topological space is a Kan complex. (Hint: use the previous part.)

(c) Define a simplicial map $\eta_X : X \to \mathrm{Sing}\,|X|$.

(d) Say that a simplicial map is a *weak equivalence* if its induced map on geometric realizations is a homotopy equivalence. A standard result in simplicial homotopy theory, sometimes called *Milnor's theorem*, states that η_X is a weak equivalence for any X.

Explain why this fact, together with part (b), justifies the tagline "spaces mod homotopy are the same as Kan complexes."

There is a much more elegant way to state this: there is an equivalence of ∞-categories from the ∞-category of topological spaces to the ∞-category of Kan complexes. Unfortunately, proving this result requires more simplicial homotopy theory than we have time for, so this will have to suffice.

Tomorrow, we will show that ∞-groupoids are also the same as Kan complexes, completing the proof of the homotopy hypothesis.

Problem 5 (optional, requires a bit more category theory background)**.** Recall that $N : \textsc{Cat} \to \textsc{sSet}$ can be defined on objects by $(NC)_n = \textsc{Cat}_1([n], C)$, and $\mathrm{Sing} : \textsc{Top} \to \textsc{sSet}$ can be defined by $(\mathrm{Sing}\, X)_n = \textsc{Top}_1(\Delta^n, X)$. In this exercise we will explore the similarity between functors defined in this way.

(a) Let C be any (small) category, and let $\hat{C}$ be the category of functors $C^{\mathrm{op}} \to \textsc{Set}$. Define a functor $y : C \to \hat{C}$, called the *Yoneda embedding*, which sends c to the functor $C_1(-, c)$.

(b) Let $\mathcal{E}$ be any other category, and let $F : C \to \mathcal{E}$ be a functor. Define a functor $G : \mathcal{E} \to \hat{C}$, called the *restricted Yoneda embedding* or the *F-relative nerve*, which sends e to the functor $C_1(F-, e)$.

(c) Suppose that $\mathcal{E}$ has coproducts and some notion of quotient ("coequalizers"). Define a functor $\hat{F} : \hat{C} \to \mathcal{E}$, called the *Yoneda extension* or the *F-relative realization*, which sends a functor $P : C^{\mathrm{op}} \to \textsc{Set}$ to the "coend"

$$\left(\coprod_{c \in C, x \in Pc} Fc \right) /\!\sim .$$

To define the relation $\sim$, we instead give two maps into this coproduct, whose images $\sim$ should identify:

$$\coprod_{f : c \to c' \in C,\, x \in Pc'} Fc \rightrightarrows \coprod_{c \in C,\, x \in Pc} Fc.$$

The top map sends the copy of Fc indexed by $(f, x \in Pc')$ via the identity to the copy indexed by $(c, Pf(x) \in Pc)$. The bottom map sends the copy of Fc indexed by $(f, x \in Pc')$ to the copy of Fc' indexed by $(c', x \in Pc')$ via the map $Ff : Fc \to Fc'$. Explain how this reduces to the geometric realization in the case that $C = \Delta$, $\mathcal{E} = \textsc{Top}$, and F sends $[n]$ to $|\Delta^n|$.

This is a construction of the left Kan extension *of F along y; in fact Kan invented Kan extensions in order to study this situation. Here are some facts, which I don't dare ask you to prove without significantly more category theory. First, there is a natural isomorphism $\hat{F} \circ y \cong F$. Second, F is unique (up to isomorphism) with this property; another way to say this is that $\hat{C}$ is the* free cocompletion *of C. Third, there is an adjunction $\hat{F} \dashv G$.*

(d) Consider the functor $\Delta \to \textsc{Cat}$ which sends the ordinal $[n]$ to the category $[n]$. What is the Yoneda extension of this functor?