# Oops All Algebra: Homework II

Mathcamp 2025

**Problem 1** (recommended)**.** In full glory, a *simplicial set* $X$ consists of sets $\{X_k : k \geq 0\}$, whose elements are called $k$*-simplices*, together with maps

$$d_i^k : X_k \longrightarrow X_{k-1}, \quad s_i^k : X_k \longrightarrow X_{k+1}, \quad (0 \leq i \leq k),$$

called the *face maps* and *degeneracy maps*, respectively. These maps satisfy the five *simplicial identities*:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && i < j \\ s_i s_j &= s_j s_{i-1} && i > j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \\ \mathrm{id} & \\ s_j d_{i-1} & \end{cases} && \begin{aligned} &i < j \\ &i \in \{j, j+1\} \\ &i > j+1. \end{aligned} \end{aligned}$$

Explain each of the five identities in terms of our intuitive picture of simplicial sets as collections of simplices with orderings on their vertices.

**Problem 2** (recommended)**.** Convince yourself that there is a bijection

$$\{\text{simplicial maps } \Delta^m \longrightarrow \Delta^n\} \cong \{\text{monotone maps } [m] \longrightarrow [n]\}$$

which sends a simplicial map $f$ to its action on vertices $f_0$.

**Problem 3** ((a)-(c) are recommended)**.** Recall from class that the *nerve* of a category $C$ is a simplicial set $NC$ with

$$(NC)_k = \left\{ x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_k} x_k \in C \right\}.$$

(a) Define the face and degeneracy maps of $NC$, and convince yourself that they satisfy the simplicial identities.

(b) Let $[n]$ be the category whose objects are elements of $[n] = \{0, \dots, n\}$ (yes, I know we have the same notation for the category and the set, sorry…) with a unique morphism $i \longrightarrow j$ whenever $i \leq j$. Convince yourself that $N[n] \cong \Delta^n$ in the category $\mathrm{sS{\small ET}}$.

(c) Convince yourself that there is a bijection $(NC)_k \cong \mathrm{C{\small AT}}_1([k], C)$, where the latter is the set of functors from $[k]$ to $C$.

(d) Convince yourself that $N$ extends to a functor $\mathrm{C{\small AT}} \longrightarrow \mathrm{sS{\small ET}}$. In class, we defined the action of this functor on morphisms in $\mathrm{C{\small AT}}$; you need to understand why this action yields a simplicial map and why it preserves composition and the identities.

**Problem 4** (optional, Homotopy Hypothesis I)**.** Over the next three days, we will formally state and prove a version of the homotopy hypothesis, which says "spaces are $\infty$-groupoids". Here is part one of that process.

Let $|\Delta^n|$ be the *topological* $n$*-simplex*, defined as the subspace

$$\{(x_0, \dots, x_n) : x_i \geq 0, \textstyle\sum x_i = 1\} \subseteq \mathbb{R}^{n+1}.$$

(a) For $X$ a topological space, define the *singular simplicial set* $\operatorname{Sing} X$ by

$$(\operatorname{Sing} X)_n = \mathrm{T{\small OP}}_1(|\Delta^n|, X).$$

Define the face and degeneracy maps, and convince yourself that they satisfy the simplicial identities. (Hint: define first the appropriate maps $d_i^n : |\Delta^n| \longrightarrow |\Delta^{n-1}|$, and similarly for the degeneracies.)

(b) Convince youself that Sing extends to a functor $\textsc{Top} \to \textsc{sSet}$.

(c) For $X$ a simplicial set, define the *geometric realization* $|X|$ as the quotient space

$$\left(\coprod_{k \geq 0} |\Delta^k| \times X_k\right) /\sim,$$

where $\sim$ is the equivalence relation generated by:

- identifying the $i$th face of $\{x\} \times |\Delta^k|$ with $\{d_i x\} \times |\Delta^{k-1}|$, and
- identifying the degenerate $k$-simplex $\{s_i x\} \times |\Delta^k|$ with the $k-1$ simplex $\{x\} \times |\Delta^{k-1}|$.

The idea is that we first make a bunch of disjoint copies of the topological $k$-simplex for each $k$-simplex in the simplicial set $X$, and then identify these disjoint simplices according to the structure maps of $X$. Convince yourself that this extends to a functor $\textsc{sSet} \to \textsc{Top}$.

(d) Convince yourself that the geometric realization of $\Delta^n$ is homeomorphic to $|\Delta^n|$, as the notation suggests.

In the next homework, we will continue the story by characterizing the image of the singular simplicial set functor, i.e. giving conditions on a simplicial set $X$ which are equivalent to the existence of a space $Y$ so that $|Y| \cong X$. In ill turn out that, when we interpret certain simplicial sets as categories, this characterization is equivalent to the interpretation of $X$ as an $\infty$-groupoid.

**Problem 5** (optional, Dold-Kan I)**.** A *simplicial abelian group* is a simplicial set $A$ so that each $A_k$ is an abelian group and the structure maps are all homomorphisms. In this problem, we will begin to prove the *Dold-Kan correspondence*, which says in a certain sense that simplicial abelian groups and chain complexes are the same thing. (Recall from day one that a chain complex $X_\bullet = (X_0 \xrightarrow{d^0} X_1 \xrightarrow{d^1} \dots)$ is a sequence of abelian groups and homomorphisms so that $d^{k+1} \circ d^k = 0$.)

(a) First let $A$ be a simplicial abelian group. Define a chain complex $(MA)_\bullet$, called the *normalized Moore complex* of $A$, via $(MA)_k = \bigcap_{i=1}^k \ker d_i^k$ for each $k \geq 0$. The differential $d^k : (MA)_k \to (MA)_{k+1}$ is given by the restriction of the remaining face map $d_0^k$. Convince yourself $d^k$ is well-defined and that $(MA)_\bullet$ is a chain complex, i.e. that $d^{k+1} \circ d^k = 0$.

(b) Let $\textsc{sAb}$ be the category whose objects are simplicial abelian groups and whose morphisms are simplicial maps, all of whose levelwise components are homomorphisms. Let $\textsc{Ch}_{\geq 0}(\textsc{Ab})$ be the category of chain complexes. Convince yourself that $M$ extends to a functor $\textsc{sAb} \to \textsc{Ch}_{\geq 0}(\textsc{Ab})$.

(c) Conversely, let $X_\bullet$ be a chain complex. Define a simplicial abelian group $\Gamma X$, the *simplicial nerve* of $X_\bullet$, by

$$(\Gamma X)_k = \bigoplus_{[k] \twoheadrightarrow [j]} X_j,$$

where the sum is over all $j$ and all monotone surjections $[k] \twoheadrightarrow [j]$. For each $0 \leq i \leq k$, let $\delta_i^k : [k-1] \hookrightarrow [k]$ be the monotone injection which misses $i$ and let $\sigma_i^k : [k+1] \twoheadrightarrow [k]$ be the monotone surjection which repeats $i$. The degeneracy maps $s_i^k : (\Gamma X)_k \to (\Gamma X)_{k+1}$ are given by mapping the summand indexed by a map $[k] \twoheadrightarrow [j]$ into the summand indexed by the composite $[k+1] \xrightarrow{\sigma_i^k} [k] \twoheadrightarrow [j]$ via the identity.

The face maps $d_i^k : (\Gamma X)_{k-1} \to (\Gamma X)_k$ act on the summand indexed by $\phi : [k] \twoheadrightarrow [j]$ as follows. If $\phi \circ \delta_i^k : [k-1] \to [j]$ is not surjective, then it misses at most one element $\ell \in [j]$, so it factors as $\phi \circ \delta_i^k = \delta_\ell^j \circ \psi$ for some surjection $\psi : [k-1] \twoheadrightarrow [j-1]$. Now for a copy of $X_\ell$ indexed by a map $\psi : [k] \twoheadrightarrow [\ell]$, write an element $x \in X_\ell$ as $x_\psi$. We can now define:

$$d_i^k(x_\phi) = \begin{cases} x_{\phi \circ \delta_i^k} & \text{if } \phi \circ \delta_i^k \text{ is surjective} \\ (-1)^\ell d(x)_\psi & \text{if } \phi \circ \delta_i^k = \delta_\ell^j \circ \psi \text{ with } \psi : [k-1] \twoheadrightarrow [j-1] \text{ surjective.} \end{cases}$$

Convince yourself that $\Gamma X$ is a simplicial abelian group, i.e. that these maps satisfy the simplicial identities.

(d) Convince yourself that $\Gamma$ extends to a functor $\textsc{Ch}_{\geq 0}(\textsc{Ab}) \to \textsc{sAb}$.

In the next homework, we will have a little bit more knowledge about simplicial sets and will be able to show that these functors form an *equivalence of categories* between $\textsc{sAb}$ and $\textsc{Ch}_{\geq 0}(\textsc{Ab})$, completing the proof.