Oops All Algebra: Homework I

Mathcamp 2025

Small lie in class today: in the chain complexes example, we need to take the tensor product rather than the ordinary product with the interval to get a good homotopy theory. This is given by

$$(X_\bullet \otimes Y_\bullet)_n = \bigoplus_{i+j=n} X_i \otimes Y_j.$$

Sorry!!

Problem 1. Suppose that $H : X \times I^n \to Y$ is a $(n+1)$-cell in the infinity category of spaces, between the n-cells $H(-,0), H(-,1) : X \times I^{n-1} \to Y$. Define an inverse homotopy $H^{-1} : X \times I^n \to Y$ between $H(-,1)$ and $H(-,0)$, and convince yourself that there is an $(n+2)$-cell between $H^{-1} \circ^{n-1} H$ and the identity homotopy $H(-,t) = H(-,0)$.

Problem 2. A *strict 2-category* consists of the following data:

- a set of 0-cells C_0;
- for each pairs of objects $x, y \in C_0$, a category $C_1(x,y)$, whose objects are called 1-cells and whose morphisms are called 2-cells;
- for each triple of 0-cells $x, y, z \in C_0$, a functor

  $$\circ_{x,y,z} : C_1(x,y) \times C_1(y,z) \to C_1(x,z);$$

  and
- for each 0-cell $x \in C_0$, a distinguished object $1_x \in C_1(x,x)$;

so that composition is (strictly) associative and unital.

(Another word for this is a *category enriched in categories.*)

(a) Convince yourself that (spaces, continuous maps, homotopy classes of homotopies) form a strict 2-category.

(b) If you know what a natural transformation is, convince yourself that (categories, functors, natural transformations) form a strict 2-category.

(c) Identify two different ways to get a single 2-cell out of the diagram

$$\begin{tikzcd}
x \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\beta"] & y \arrow[r, bend left=60, "f'"] \arrow[r, "g'"] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=4, "\Downarrow\beta'"] & z.
\end{tikzcd}$$

Why do these two composites agree?

Problem 3. A *strict n-category* (for $n \in \mathbb{N} \cup \{\infty\}$) is a sequence of sets and functions

$$C_0 \leftleftarrows C_1 \leftleftarrows \cdots \leftleftarrows C_n,$$

thought of as sequences of source and target maps, so that $C_j \leftleftarrows C_k$ has the additional structure of a category for every $j < k \le n$, and so that each $C_j \leftleftarrows C_k \leftleftarrows C_\ell$ is a strict 2-category for every $j < k < \ell \le n$.

(a) If you know what a natural transformation is, convince yourself that there is a strict $n+1$-category of strict n-categories.

(b) Explain why (spaces, continuous maps, homotopies, homotopy classes of homotopies between homotopies) fails to be a strict 3-category.

Problem 4. Let $I = [0,1]$ be the topological unit interval. In this problem, we will prove a universal property of I due to [Lei10] that explains its importance in homotopy theory.

(a) Suppose X and Y are spaces with points $x \in X$, $y \in Y$, which correspond to maps $x : * \to X$ and $y : * \to Y$. Form the *wedge sum* $X \vee Y = (X,x) \vee (Y,y)$ of X and Y by taking the disjoint union $X \vee Y$ and identifying x and y. Define continuous maps $\iota_X : X \hookrightarrow X \vee Y$ and $\iota_Y : Y \hookrightarrow X \vee Y$. Then show that $X \vee Y$ has the following universal property: for any other space Z equipped with maps $f : X \to Z$ and $g : Y \to Z$ so that $f(x) = g(y)$, there is a unique map $h : X \vee Y \to Z$ making the diagram

$$\begin{tikzcd}
 & & Z \\
X \arrow[r, "\iota_X"] \arrow[urr, bend left, "f"] & X \vee Y \arrow[ur, dashed, "\exists! h"] & \\
* \arrow[u, "x"] \arrow[r, "y"'] & Y \arrow[u, "\iota_Y"'] \arrow[uur, bend right, "g"'] &
\end{tikzcd}$$

commute. (Categorically, this is called the pushout of ι_X and ι_Y, or the coproduct in the category of pointed spaces.)

(b) A *bipointed space* is a space X equipped with two points $x_0 \neq x_1$. A *map of bipointed spaces* from (X, x_0, x_1) to (Y, y_0, y_1) is a continuous map $f : X \to Y$ so that $f(x_0) = y_0$ and $f(x_1) = y_1$. Given two bipointed spaces (X, x_0, x_1) and (Y, y_0, y_1), define the *bipointed wedge sum* $X \vee Y$ to be the wedge sum $(X, x_1) \vee (Y, y_0)$; this is a bipointed space with the points x_0 and y_1.

Treat the interval I as a bipointed space $(I, 0, 1)$. Define a map of bipointed spaces $I \to I \vee I$. Show that this map is an isomorphism, and explain how this fact allows us to define composition of homotopies.

(c) Show that the interval is the universal such object, in following sense. Suppose that X is any other bipointed Hausdorff space equipped with a map of bipointed spaces $X \to X \vee X$. Show that there is a unique map of bipointed spaces $X \to I$ such that

$$\begin{tikzcd}
X \arrow[r] \arrow[d, dashed, "\exists!"'] & X \vee X \arrow[d, dashed] \\
I \arrow[r] & I \vee I
\end{tikzcd}$$

commutes. (How do you define the map $X \vee X \to I \vee I$?)

(Hint: first show that analogous fact in the category of sets, forgetting the topologies.

(d) Say that an *interval object* in a category C is any object I equipped with the same universal property as the interval (spell out what this means in purely categorical terms). Show that the map $I \to I \vee I$ for any such I is always an isomorphism.

(e) Show that if I and J are interval objects in C, then they are isomorphic.

(f) (Probably very difficult.) Show that $I_\bullet = (\mathbb{Z} \xrightarrow{(\mathrm{id}, -\mathrm{id})} \mathbb{Z} \oplus \mathbb{Z} \to 0 \to 0 \to \dots)$ is the interval object in the category of chain complexes of abelian groups.

References

[Lei10] Tom Leinster. *A general theory of self-similarity*. 2010. arXiv: 1010.4474 [math.CT]. URL: https://arxiv.org/abs/1010.4474.