Extra-Stretchy Rubber-Sheet Geometry: Day IV Homework

Mathcamp 2025

Problem 1 (recommended)**.** Describe the universal cover of the torus, the annulus, and the "wedge of three spheres" (like a figure eight but with three rather than two circles connected at a point).

Problem 2 (recommended)**.** Argue that, if X and Y are spaces, a loop in $X \times Y$ is the same data as a pair of loops, one in X and one in Y. (One way to do this is to give a bijection from the set of loops in $X \times Y$ to the set of pairs of loops in X and loops in Y.) If you know what the direct product of groups is, argue further that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Optional problems really are optional!! *Just here in case you find them fun / want some challenges, not at all needed for what we're doing in class.*

Problem 3 (optional)**.** In the following exercises, we will be more precise about the properties of loop homotopy.

(a) Argue that, if p, p', q, q' are loops in a space X with $p \simeq p'$ and $q \simeq q'$, then $p \bullet q \simeq p' \bullet q'$. (Why does this fact matter to us?)

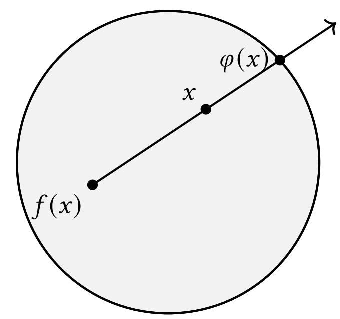
(b) Argue that loop homotopy forms an *equivalence relation*. This means that it is:

- *reflexive*: every loop p is homotopic to itself;
- *symmetric*: if p is homotopic to q, then q is homotopic to p;
- *transitive*: if p is homotopic to q and q is homotopic to r, then p is homotopic to r.

(c) (If you know what a group isomorphism is.) Let x, x' be points in a topological space X, and suppose there is a path between them. Show that there is an isomorphism $\pi_1(X, x) \cong \pi_1(X, x')$.

Problem 4 (optional, if you know what a group homomorphism is)**.** Let $f : X \to Y$ be a continuous map between two topological spaces. Argue that f induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ on the fundamental groups of X and Y. Prove further that if $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then $(g \circ f)_* = g_* \circ f_*$; and that this construction sends identities to identities, in that $(1_X)_* = 1_{\pi_1(X)}$. (All this is to say that π_1 is a *functor*.)

Problem 5 (optional, difficult)**.** In this problem, we will prove the *Brouwer fixed-point theorem*, which says that any continuous map $f : D^2 \to D^2$ has at least one fixed-point, i.e. a point $x \in D^2$ such that $f(x) = x$.



(a) Suppose, towards a contradiction, that there is a map $f : D^2 \to D^2$, so that $f(x) \neq x$ for all $x \in D^2$. Fix for a moment some $x \in D^2$. Since two distinct points determine a ray, we can consider the unique ray starting at $f(x)$ and passing through x. Let $\varphi(x)$ be the point where this ray intersects the boundary of the disk, i.e. the circle. Then φ is a continuous map $D^2 \to S^1$. Argue that φ acts by the identity on points on the boundary circle, i.e. $\varphi(x) = x$ for all $x \in S^1$.

(b) Let $i : S^1 \hookrightarrow D^2$ be the map that includes the circle as the boundary of the disk. In the previous part, we defined a map $\varphi : D^2 \to S^1$ so that $\varphi \circ i = 1_{S^1}$. According to Problem 4, these maps induce group homomorphisms $\varphi_* : \pi_1(D^2) \to \pi_1(S^1)$ and $i_* : \pi_1(S^1) \to \pi_1(D^2)$ with $\varphi_* \circ i_* = 1_{\pi_1(S^1)}$. Using our computations of $\pi_1(S^1)$ and $\pi_1(D^2)$, argue that this is an impossibility, hence the assumption of the previous part must be false.