Extra-Stretchy Rubber-Sheet Geometry: Day III Homework

Mathcamp 2025

Today we defined the notion of a homotopy of two loops in a topological space X. If there is a homotopy between two loops γ and δ, we write $\gamma \simeq \delta$. We also discussed the concatenation of two loops, which we write $\gamma \bullet \delta$.

Problem 1 (recommended)**.** In this problem I'm not asking you to write proofs of these things, although if you want to you are welcome; instead I'm just asking you to draw a picture and think about the properties of composition of loops. Let x_0 be a point in a space X.

(a) Convince yourself that concatenation of loops has a *unit*, in that there is a loop c based at x_0 such that for any loop γ based at x_0, $c \bullet \gamma \simeq \gamma \bullet c \simeq \gamma$. (What is the loop c?)

(b) Convince yourself that concatenation of loops is *associative*, in that for any loops γ, δ, and ϵ all based at x_0, we have

$$(\gamma \bullet \delta) \bullet \epsilon \simeq \gamma \bullet (\delta \bullet \epsilon).$$

(c) If γ is a loop in X based at x_0, let $\bar{\gamma}$ be the loop that traverses the same path as γ but in the opposite direction. Convince yourself that $\bar{\gamma}$ is the *inverse* of γ, in that $\gamma \bullet \bar{\gamma} \simeq c$. (This amounts to explaining how to drag $\gamma \bullet \bar{\gamma}$ into c without moving the basepoint.)

Optional problems really are optional!! *Just here in case you find them fun / want some challenges, not at all needed for what we're doing in class.*

Problem 2 (optional)**.** For each piece of Problem 1, give a precise proof of the relevant property, using the algebraic definition of a homotopy discussed in class as a map $H : S^1 \times I \to X$.

Problem 3 (optional)**.** Define a continuous surjective map $f : \mathbb{R} \to S^1$ from the real line to the circle. (A map $f : A \to B$ is *surjective* if it hits every point in B.) For each point p in the circle, describe the set of real numbers which map to p under f (this set is called the *preimage* or *fiber* of p under f, and often written $f^{-1}(p)$.)

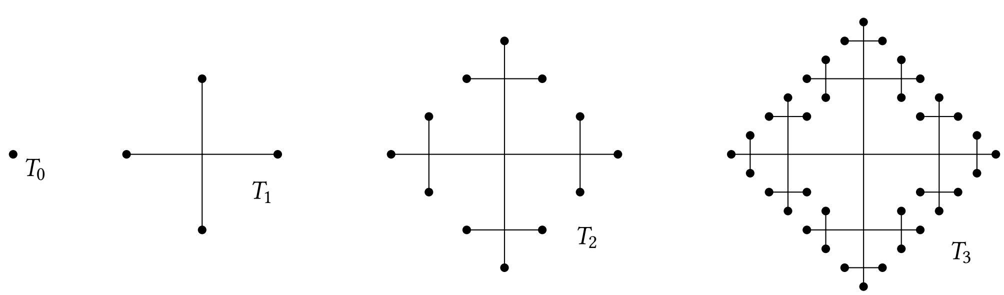
Problem 4 (optional)**.** Do Problem 3 again, but with a map from the "thickened line" $\mathbb{R} \times I$ to the closed annulus.

Problem 5 (optional)**.** Do Problem 3 again, but with a map from the plane $\mathbb{R}^2$ to the torus T^2.

Problem 6 (optional)**.** Consider the space T defined as follows. Start with a single point T_0, and connect four line segments to it, to obtain the space T_1. Then, for each of the four endpoints of T_1, connect three line segments to it (making a total of four counting the one already connected) to obtain the space T_2. Continue this process infinitely to obtain T.

(One name for T is the *infinite 4-valent tree.*)

Do Problem 3 again, but with a map from T to the figure-eight.



Problem 7 (optional, just a little food for thought; I don't expect you to be able to come up with a complete answer)**.** What do the spaces and maps in Problems 3 to 6 have in common?