Extra-Stretchy Rubber-Sheet Geometry: Day II Homework

Mathcamp 2025

In class, we discussed some tools to separate spaces, i.e. to prove that they are not homeomorphic. The general pattern for proving two spaces are not homeomorphic goes as follows:

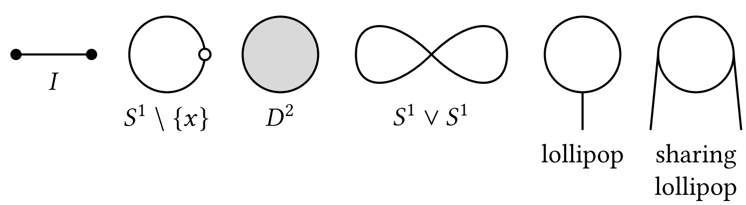
1. Find some value, $i(X)$, which can be computed for any space X;
2. Prove that i is an invariant of homeomorphism, i.e. that if $X \cong Y$, then $i(X) = i(Y)$;
3. Show that $i(X) \neq i(Y)$ for the specific spaces X and Y you are trying to prove are not homeomorphic.

Some invariants we discussed included path-connectedness, cardinality (the number of points), the IELPs (points locally homeomorphic to the endpoint of an interval), path-cut-points (points such that removing them gives back a non-path-connected space), and non-path-cut-points (points such that removing them gives back a path-connected space).

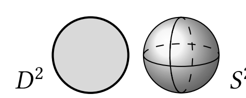
- The double interval $[-2,-1] \cup [1,2]$ is not homeomorphic to the interval I, since the former is not path-connected, while the latter is.
- The closed interval I is not homeomorphic to the open interval $(0,1)$, since the former has two non-path-cut-points (the endpoints), while the latter has none.
- The real line $\mathbb{R}$ is not homeomorphic to the plane $\mathbb{R}^2$, since every point of the former is a path-cut point, while no points of the latter are.

Problem 1 (recommended)**.** Argue that none of the following spaces are homeomorphic to each other:

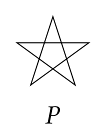
- the closed interval I,
- the punctured circle $S^1 \setminus \{x\}$,
- the closed disk D^2,
- the figure eight $S^1 \vee S^1$,
- the hollow lollipop,
- the hollow "sharing" (two-stick) lollipop,



Problem 2 (recommended)**.** Explain why none of the aforementioned invariants can distinguish the sphere S^2 and the closed disk D^2. (Hard but fun: distinguish them anyway!)

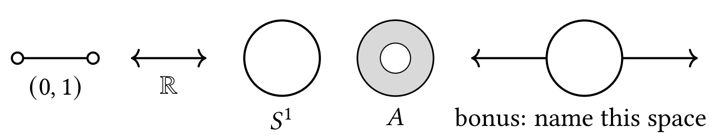


Problem 3 (optional)**.** Prove that the pentagram P is not homeomorphic to the circle S^1. (Notice that both are path-connected and that neither has any path-cut points. You need some strengthened invariant to distinguish them, but it is possible to find a "souped-up" version of path-cut points to do so. Definitely talk to each other and/or to me!)



Problem 4 (optional)**.** For each of the following spaces, identify the number of path-cut and non-path-cut-points (either of which may be infinite).

- the open interval $(0,1)$,
- the real line $\mathbb{R}$,
- the circle S^1,
- the annulus A,
- the "circle with two rays" displayed at far right.



Problem 5 (optional)**.** For each natural number $n \geq 0$, find a space X_n with exactly n path-cut points, and a space Y_n with exactly n non-path-cut points. Do this also for $n = \infty$.

Problem 6 (optional)**.** Brainstorm as many ideas as you can for potential homeomorphism-invariants. If you want you can try to prove that they are invariants, but just writing down half-baked ideas is perfect.