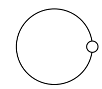
Extra-Stretchy Rubber-Sheet Geometry: Day I Homework

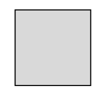
Mathcamp 2025

Today, we discussed the notion of homeomorphism—relating two rubber-sheet spaces to each other via a sequence of stretches-and-compressions. Our goal in this homework is to continue building intuition for this notion.

Problem 1 (recommended)**.** Let X be the *punctured circle*, which is formed by cutting a point out of the circle, pictured right. (Choose any point you prefer to cut out—topologically it's all the same!) Argue (by drawing a picture or giving a formula) that X is homeomorphic to the real line $\mathbb{R}$. (Equivalently, you may argue that it is homeormorphic to an open interval (a, b), since any open interval is homeomorphic to the real line.)



Problem 2 (recommended)**.** In class, we defined the closed disk D^2. Let R be the closed unit rectangle (ok yes it's just a square…topologists are weird), together with its interior, as pictured at right. Argue that D^2 and R are homeomorphic.

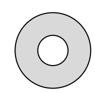


Problem 3 (recommended)**.** Let T^1 be the "torus", discussed in class—it is the surface of a donut. Let X be a coffee cup. We can think of X as consisting of the base and curved side of a cylinder, together with a handle. Convince yourself that T^1 and X are homeomorphic. (It may help to look up a video of this—there is a nice one called "Coffee Cup Donut" by Jim Fowler on youtube.)

Problem 4 (optional)**.** In class we briefly discussed the following non-obvious fact: any homeomorphism between "sufficiently nice" spaces, including those we discussed in class, can be produced via an sequence of stretch-and-compressions. Meanwhile, the line $x = 0$ and the line $y = 0$ are homeomorphic, essentially via rotating the plane. Explain how to realize this homeomorphism as a sequence of stretch-and-compressions.

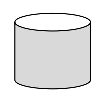
Problem 5 (optional)**.** Let A be the *closed annulus* in $\mathbb{R}^2$,

$$A = \{x \in \mathbb{R}^2 : 1 \leq ||x|| \leq 2\}.$$



Let C be the curved face of a cylinder,

$$C = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in S^1, -1 \leq z \leq 1\}.$$



(Do not worry if you feel more comfortable with the pictures than the symbols, or vice vera—they're both there for a reason! One of the great things about this subject is you can often pick whether you prefer to think symbolically or geometrically.)

In class, we argued that A and C are homeomorphic. Do you think either is homeomorphic to the circle S^1?

Problem 6 (optional)**.** Argue that homeomorphism forms an *equivalence relation*. This means that it is:

- *reflexive*: every space X is homeomorphic to itself;
- *symmetric*: if X is homeomorphic to Y, then Y is homeomorphic to X; and
- *transitive*: if X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z.

Hint: to do symmetry, for instance, you might take some strategy for stretching and compression X to Y, and argue that you can "reverse" it to get a strategy for stretching and compressing Y to X. If you prefer, you could also do this using the formal definition we discussed at the end of class.

Problem 7 (optional)**.** You can often write a formula for homeomorphisms. For instance, the homeomorphism from the open interval $(-1, 1)$ to the real line $\mathbb{R}$ can be given in one direction by the formula $f(x) = \frac{x}{1-|x|}$.

If you know what these words mean, prove that this is a continuous bijection. Then choose any of the homeomorphisms we discussed in class, or those from the problems above, and write a formula for a specific homeomorphism. Repeat as many times as you like.