# Category Theory from Scratch: Day I Homework

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In class today, we started a table of various mathematical domains and various constructions within them. We then gave a definition of a category, which will provide a common language for talking about these domains.

**Definition 1.** A *category* $C$ consists of:

- a set of *objects*;
- for each pair of objects $x, y$, a set of *morphisms*;
- for each object $x$, an *identity morphism* $x \xrightarrow{1_x} x$;
- for each pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$, a *composite morphism* $x \xrightarrow{g \circ f} z$;

so that:

- composition is *associative*: for any triple of morphisms $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, we have
$$(h \circ g) \circ f = h \circ (g \circ f);$$
- the identity is a *unit*: for any morphism $x \xrightarrow{f} y$, we have
$$f = 1_y \circ f = f \circ 1_x.$$

Our goal in this homework is to build some intuition for this definition, mostly by looking at lots of examples. we recommend doing Problems 1 to 3, and at least one of Problems 4 to 8; the rest are optional.

**Problem 1.** Prove that the category $\textsc{Set}$ of sets—whose objects are sets, morphisms are functions, identities are the identity functions, and composiiton is ordinary function composition—satisfies the definition of a category.

**Problem 2.** Choose at least one row on our table, and prove that the choices of objects and arrows given there form a category. For instance:

- Prove that the category $\textsc{Grp}$ of groups and group homomorphisms is a category.
- Prove that the category $\textsc{Top}$ of topological spaces and continuous maps is a category.
- Prove that the category $\textsc{Vect}_{\mathbb{C}}$ of complex vector spaces and linear maps is a category. (For fun: do this for any field $\Bbbk$.)
- Prove that the category $\textsc{CRing}$ of commutative rings and ring homomorphisms is a category.

*Think about what you really need to prove for these examples. How much follows from the case of* $\textsc{Set}$*, since each of these objects and morphisms has a set and a set-function underlying it?*

**Problem 3.** Given a category $C$, define a new category $C^{\mathrm{op}}$ whose objects are the same as $C$, but whose morphisms $x \to y$ are the morphisms $y \to x$ in $C$. Define composition and the identities, and prove that $C^{\mathrm{op}}$ is a category.

*The category* $C^{\mathrm{op}}$ *is called the* opposite category *of* $C$.

**Problem 4.** Let $A$ be a set. Define a category whose objects are the elements of $A$, and with a unique morphism $x \to y$ if and only if $x = y$. Define composition and the identities, and prove that this defines a category.

*This category is called the* discrete category *on* $A$.

**Problem 5.** Let $n$ be a natural number. Define a category $\underline{n}$ whose objects are the numbers $0, \ldots, n-1$, and with a unique morphism $i \to j$ if and only if $i \le j$. Define composition and the identities, and prove that $\underline{n}$ is a category.

*The category* $\underline{n}$ *is called the* ordinal category *associated to* $n$.

**Problem 6.** Let $P$ be a *preordered set*. This means that $P$ consists of a set, together with a relation $\leq$ on that set which is:

- *reflexive*: for any $x \in P$, we have $x \leq x$;
- *transitive*: for any $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

Define a category, $|P|$, whose objects are the elements of $P$ and with a unique morphism $x \to y$ if and only if $x \leq y$. Define composition and the identities, and prove that $|P|$ is a category.

*I do not know of a separate name for $|P|$, because it is somewhat common to view $P$ as identical with its associated category $|P|$. Indeed, it is a fun exercise to show that any category $C$ with at most one morphism between any pair of objects determines a preordered set.*

**Problem 7.** Let $M$ be a *monoid*. This means that $M$ consists of a set, together with a binary multiplication and an identity $e \in M$ so that:

- multiplication is associative: for any $x, y, z \in M$, we have
$$(xy)z = x(yz);$$
- $e$ is a unit: for any $x \in M$, we have
$$ex = x = xe.$$

Define a category, $BM$, with a unique object $*$ and the morphisms $* \to *$ the elements of $M$. Define composition and the identities, and prove that $BM$ is a category.

*The category $BM$ is sometimes called the* delooping *of the monoid $M$.*

**Problem 8.** Let $G$ be a directed graph. This means that $G$ consists of a set $V$ of vertices and a set $E \subseteq V \times V$ of edges; an element $(v, w) \in E$ represents an edge from $v$ to $w$ in $G$. A *path of length $n$* in $G$ between $v_1$ and $v_n$ consists of a sequence $(v_0, v_1), (v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ of edges, which may repeat. (In the case $n = 0$, we have the *empty path*.)

Define a category, $\mathcal{F}G$, whose objects are the vertices of $G$ and whose morphisms $v \to w$ are the paths between $v$ and $w$ in $G$. Define composition and the identities, and prove that $\mathcal{F}G$ is a category.

*The category $\mathcal{F}G$ is called the* free *or* path category *associated to the graph $G$.*

**Problem 9.** Given a category $C$ and an object $x$, define a new category $C/x$ as follows:

- The objects of $C/x$ are pairs $(y, f)$, where $y$ is an object in $C$ and $f : y \to x$ is a morphism in $C$.
- The morphisms between $(y, f)$ and $(z, g)$ in $C/x$ are the morphisms $h : y \to z$ in $C$ so that $g \circ h = f$.

Define composition and the identities, and prove that $C/x$ is a category.

*The category $C/x$ is called the* slice *or* over category *of $C$ at $x$. You can also define an* under category *$x/C$—it is a fun exercise to figure out how to do so!*