Category Theory from Scratch: Sums Worksheet

Mathcamp 2025 - Della Hendrickson & Riley Shahar

We have now defined a *sum*:

Definition 1. Let x and y be objects in a category C, and let z be an object with morphisms $\iota_x : x \to z$ and $\iota_y : y \to z$. We say (z, ι_x, ι_y) is a *sum* of x and y if, for any object w with morphisms $f : x \to w$ and $g : y \to w$, there exists a unique morphism $h : z \to w$ such that $h \circ \iota_x = f$ and $h \circ \iota_y = g$.

$$\begin{tikzcd}
x \arrow[dr, "\iota_x"'] \arrow[drr, bend left, "f"] & & \\
& z \arrow[r, dashed, "\exists! h"] & w \\
y \arrow[ur, "\iota_y"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Our goal in this worksheet is to work out some properties of this definition.

Problem 1. Describe what a sum is in as many categories as you can.

Problem 2. In this exercise, we will prove that sums are unique up to unique isomorphism.

$$\begin{tikzcd}
x \arrow[d, "\iota_x"'] \arrow[dr, "\iota'_x"] & \\
z \arrow[r, "\sim", "f"'] & z'
\end{tikzcd}$$

(a) Prove that, if (z, ι_x, ι_y) and (z', ι'_x, ι'_y) are sums of x and y, then there is an isomorphism $f : z \to z'$ such that $\iota'_x = f \circ \iota_x$ and $\iota'_y = f \circ \iota_y$. We are therefore justified in writing z or z' as $x + y$.

(b) Prove that if f and g are isomorphisms satisfying this property, then $f = g$.

Problem 3 (optional)**.** In this exercise, we will prove that the sum is commutative.

(a) Prove that, for objects x and y, there is an isomorphism

$$\gamma_{x,y} : x + y \xrightarrow{\sim} y + x.$$

(b) Prove that $\gamma_{x,y}$ commutes with the inclusions, in that

$$\gamma_{x,y} \circ \iota_x = \iota_x \quad \text{and} \quad \gamma_{x,y} \circ \iota_y = \iota_y. \qquad (1)$$

$$\begin{tikzcd}
x \arrow[d, "\iota_x"'] \arrow[dr, "\iota_x"] & \\
x + y \arrow[r, "\gamma_{x,y}"'] & y + x
\end{tikzcd}$$

Warning: the ι_x on the left is a different map from the ι_x on the right! The former is inclusion from $y + x$, while the latter is inclusion from $x + y$.

(c) Prove that $\gamma_{x,y}$ is the unique isomorphism satisfying Eq. (1).

Problem 4 (optional)**.** In this exercise, we will prove that the sum is associative. We are thus justified in writing either $(x + y) + z$ and $x + (y + z)$ as $x + y + z$.

(a) Prove that, for objects x, y, and z, there is an isomorphism

$$\alpha_{x,y,z} : (x + y) + z \xrightarrow{\sim} x + (y + z).$$

(b) Prove that $\alpha_{x,y,z}$ commutes with the inclusions, in that

$$\alpha_{x,y,z} \circ \iota_{x+y} \circ \iota_x = \iota_x, \quad \alpha_{x,y,z} \circ \iota_{x+y} \circ \iota_y = \iota_{y+z} \circ \iota_y, \quad \text{and} \quad \alpha_{x,y,z} \circ \iota_z = \iota_{y+z} \circ \iota_z. \qquad (2)$$

(c) Prove that $\gamma_{x,y}$ is the unique isomorphism satisfying Eq. (2).

Problem 5 (optional)**.** Show that, if $f : x \to x'$ and $g : y \to y'$ are morphisms, then there is an induced morphism

$$f + g : x + y \to x' + y'.$$