Category Theory from Scratch: Products Worksheet

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We have now defined a *product*:

Definition 1. Let x and y be objects in a category C, and let z be an object with morphisms $\pi_x : z \longrightarrow x$ and $\pi_y : z \longrightarrow y$. We say that (z, π_x, π_y) is a *product* of x and y if, for any object w with morphisms $f : w \longrightarrow x$ and $g : w \longrightarrow y$, there exists a unique morphism $h : w \longrightarrow z$ such that $h \circ \pi_x = f$ and $h \circ \pi_y = g$.

$$\begin{tikzcd} & & x \\ w \arrow[r, dashed, "\exists! h"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & z \arrow[ur, "\pi_x"] \arrow[dr, "\pi_y"] & \\ & & y \end{tikzcd}$$

Our goal in this worksheet is to work out some properties of this definition.

Problem 1. Describe what a product is in as many categories as you can.

Problem 2. In this exercise, we will prove that products are unique up to unique isomorphism.

$$\begin{tikzcd} z \arrow[r, "f", "\sim"'] \arrow[dr, "\pi_x"'] & z' \arrow[d, "\pi'_x"] \\ & x \end{tikzcd}$$

(a) Prove that, if (z, π_x, π_y) and (z', π'_x, π'_y) are products of x and y, then there is an isomorphism $f : z \longrightarrow z'$ such that $\pi'_x = \pi_x \circ f$ and $\pi'_y = \pi_y \circ f$. We are therefore justified in writing z or z' as $x \times y$.

(b) Prove that if f and g are isomorphisms satisfying this property, then $f = g$.

Problem 3 (optional)**.** In this exercise, we will prove that the product is commutative.

(a) Prove that, for objects x and y, there is an isomorphism

$$\gamma_{x,y} : x \times y \xrightarrow{\sim} y \times x.$$

(b) Prove that $\gamma_{x,y}$ commutes with the projections, in that

$$\begin{tikzcd} x \times y \arrow[r, "\gamma_{x,y}"] \arrow[dr, "\pi_x"'] & y \times x \arrow[d, "\pi_x"] \\ & x \end{tikzcd}$$

$$\pi_x \circ \gamma_{x,y} = \pi_x \quad \text{and} \quad \pi_y \circ \gamma_{x,y} = \pi_y \tag{1}$$

Warning: the π_x on the left is a different map from the π_x on the right! The former is projection from $y \times x$, while the latter is projection from $x \times y$.

(c) Prove that $\gamma_{x,y}$ is the unique isomorphism satisfying Eq. (1).

Problem 4 (optional)**.** In this exercise, we will prove that the product is associative. We are thus justified in writing either $(x \times y) \times z$ and $x \times (y \times z)$ as $x \times y \times z$.

(a) Prove that, for objects x, y, and z, there is an isomorphism

$$\alpha_{x,y,z} : (x \times y) \times z \xrightarrow{\sim} x \times (y \times z).$$

(b) Prove that $\alpha_{x,y,z}$ commutes with the projections, in that

$$\pi_x \circ \alpha_{x,y,z} = \pi_x \circ \pi_{x \times y}, \quad \pi_y \circ \pi_{y \times z} \circ \alpha_{x,y,z} = \pi_y \circ \pi_{x \times y}, \quad \text{and} \quad \pi_z \circ \pi_{y \times z} \circ \alpha_{x,y,z} = \pi_z. \tag{2}$$

(c) Prove that $\gamma_{x,y}$ is the unique isomorphism satisfying Eq. (2).

Problem 5 (optional)**.** Show that, if $f : x \longrightarrow x'$ and $g : y \longrightarrow y'$ are morphisms, then there is an induced morphism

$$f \times g : x \times y \longrightarrow x' \times y'.$$