Category Theory from Scratch: Monomorphisms Worksheet

Mathcamp 2025 - Della Hendrickson & Riley Shahar

We have now defined a *monomorphism*:

Definition 1. A *monomorphism* in a category C is a morphism $f : x \to y$ such that, if $g, h : z \to x$ are morphisms with $f \circ g = f \circ h$, then $g = h$.

Our goal in this worksheet is to work out some properties of this definition.

Problem 1. Describe what a monomorphism is in as many categories as you can.

Problem 2. Prove that every isomorphism is a monomorphism.

Problem 3. Suppose that $f : x \to y$ and $g : y \to z$ are monomorphisms. Prove that $g \circ f$ is a monomorphism.

Problem 4. Suppose that $f : x \to y$ and $g : y \to z$ are morphisms so that $g \circ f$ is a monomorphism. Show that f is a monomorphism.

Problem 5. If $f : x \to y$ is any morphism, we can define a function

$$\begin{aligned} f_* : C(z, x) &\to C(z, y) \\ g &\mapsto f \circ g \end{aligned}$$

via postcomposing with f. Prove that f is a monomorphism if and only if f_* is injective for every object $z \in C$.

Problem 6. We set out to define a subobject, but ended up at the notion of monomorphism. But there are many monomorphisms that might represent the same subobject—for instance, there is the map $\{a\} \hookrightarrow \{0, 1\}$ sending a to 0, and the inclusion $\{0\} \hookrightarrow \{0, 1\}$ sending 0 to 0. How could you turn this definition of monomorphism into a definition of subobject that agrees with our normal notions of subset, subgroup, etc.?

Once you have such a definition, let $\mathrm{Sub}(x)$ be the collection of subobjects of x. Show that it is partially ordered. This means that you should define a relation $\leq$ on the subobjects which is:

- *reflexive*: for any subobject $s \in \mathrm{Sub}(x)$, we have $s \leq s$;
- *antisymmetric*: if $s, t \in \mathrm{Sub}(x)$, $s \leq t$, and $t \leq s$, then $s = t$; and
- *transitive*: if $s, t, r \in \mathrm{Sub}(x)$, $s \leq t$, and $t \leq r$, then $s \leq r$.

Problem 7. Define an *epimorphism* dually to a monomorphism: an epimorphism is a morphism $f : x \to y$ such that, if $g, h : y \to z$ are morphism with $g \circ f = h \circ f$, then $g = h$. Repeat Problems 1 to 3 for epimorphisms.

Problem 8. Suppose that $f : x \to y$ and $g : y \to z$ are morphisms so that $g \circ f$ is an epimorphism. Show that g is an epimorphism.

Problem 9. If $f : x \to y$ is any morphism, we can define a function

$$\begin{aligned} f^* : C(y, z) &\to C(x, z) \\ g &\mapsto g \circ f \end{aligned}$$

via precomposing with f. Prove that f is an epimorphism if and only if f^* is injective for every object $z \in C$.