Category Theory from Scratch: Completely Optional Isomorphisms Worksheet

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We have now defined an *isomorphism*:

Definition 1. A morphism $f : x \to y$ in a category C is an *isomorphism* if there exists a morphism $g : y \to x$ in C such that

$$f \circ g = 1_y \quad \text{and} \quad g \circ f = 1_x.$$

We say that x and y are *isomorphic*, and write $x \cong y$. We call g an *inverse* of f.

Our goal in this worksheet is to work out some properties of this definition.

Problem 1. Describe what an isomorphism is in as many categories as you can.

Problem 2. Prove that, if $f : x \to y$ is an isomorphism with inverses $g, h : y \to x$, then $g = h$. We therefore may call g *the* inverse of f, and write $g = f^{-1}$.

Problem 3. Prove that being isomorphic is an equivalence relation on the objects of a category, in that it is:

- *reflexive*: for any object x, we have $x \cong x$;
- *symmetric*: if $x \cong y$, then $y \cong x$; and
- *transitive*: if $x \cong y$ and $y \cong z$, then $x \cong z$.

Problem 4. Prove that, if $f : x \to y$ is an isomorphism, then for any object z, we have bijections

$$C(x, z) \cong C(y, z) \quad \text{and} \quad C(z, x) \cong C(z, y),$$

Problem 5. Prove that, if $f : x \to y$ and $g : y \to z$ are morphisms, and two of the three of f, g, and $g \circ f$ are isomorphisms, then so is the third. This is called the *two-out-of-three property*.