Category Theory from Scratch: Initial Objects Worksheet

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We have now defined an *initial object*:

Definition 1. An *initial object* in a category C is an object $\varnothing$ such that, for any object x, there is a unique morphism $!_x : \varnothing \to x$.

Our goal in this worksheet is to work out some properties of this definition.

Problem 1. Describe what an initial object is in as many categories as you can.

Problem 2. In this exercise, we will prove that initial objects are unique up to unique isomorphism.

(a) Prove that, if $\varnothing$ and $\varnothing'$ are initial objects in C, then there is an isomorphism $f : \varnothing \to \varnothing'$.

(b) Prove that if f and g are isomorphisms $\varnothing \to \varnothing'$, then $f = g$.

Problem 3. Given a category C and an object x, define the *under category* category x/C as follows:

- The objects of x/C are pairs (y, f), where y is an object in C and $f : x \to y$ is a morphism in C.
- The morphisms between (y, f) and (z, g) in x/C are the morphisms $h : y \to z$ in C so that $h \circ f = g$.

Define composition and the identities, and prove that x/C is a category.

What is the initial object in x/C?

Problem 4 (optional)**.** In the category of *pointed groups*,

- The objects are pairs (G, x), where G is a group and $x \in G$.
- The morphisms between (G, x) and (H, y) are the group homomorphisms $f : G \to H$ such that $f(x) = f(y)$.

Define composition and the identities, and prove that this is a category.

What is the initial object in the category of pointed groups?