Category Theory from Scratch: Putting It All Together

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We now have lots of definitions:

Definition 1 (isomorphism)**.** A morphism $f : x \to y$ in a category C is an *isomorphism* if there exists an *inverse* morphism $f^{-1} : y \to x$ in C such that $f \circ f^{-1} = 1_y$ and $f^{-1} \circ f = 1_x$. When such f exists, we say that x and y are *isomorphic*, and write $x \cong y$.

Definition 2 (terminal object)**.** A *terminal object* in a category C is an object $*$ such that, for any object x, there is a unique morphism $!_x : x \to *$.

Definition 3 (initial object)**.** An *initial object* in a category C is an object $\varnothing$ such that, for any object x, there is a unique morphism $!_x : \varnothing \to x$.

Definition 4 (product)**.** Let x and y be objects in a category C, and let $x \times y$ be an object with morphisms $\pi_x : x \times y \to x$ and $\pi_y : x \times y \to y$. We say that $(x \times y, \pi_x, \pi_y)$ is a *product* of x and y if, for any object w with morphisms $f : w \to x$ and $g : w \to y$, there exists a unique morphism $h : w \to x \times y$ such that $h \circ \pi_x = f$ and $h \circ \pi_y = g$.

$$
\begin{tikzcd}
 & & x \\
w \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, dashed, "\exists! h"] & x \times y \arrow[ur, "\pi_x"'] \arrow[dr, "\pi_y"] & \\
 & & y
\end{tikzcd}
$$

Definition 5 (coproduct)**.** Let x and y be objects in a category C, and let $x + y$ be an object with morphisms $\iota_x : x \to x + y$ and $\iota_y : y \to x + y$. We say $(x + y, \iota_x, \iota_y)$ is a *coproduct*, or a *sum*, of x and y if, for any object w with morphisms $f : x \to w$ and $g : y \to w$, there exists a unique morphism $h : x + y \to w$ such that $h \circ \iota_x = f$ and $h \circ \iota_y = g$.

$$
\begin{tikzcd}
x \arrow[dr, "\iota_x"'] \arrow[drr, bend left, "f"] & & \\
 & x + y \arrow[r, dashed, "\exists! h"] & w \\
y \arrow[ur, "\iota_y"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}
$$

Definition 6 (monomorphism)**.** A *monomorphism* in a category C is a morphism $f : x \to y$ such that, if $g, h : z \to x$ are morphisms with $f \circ g = f \circ h$, then $g = h$.

Our goal in this worksheet is to combine some of these definitions.

Problem 1. Show that:

- A morphism $f : x \to y$ is an isomorphism in C if and only if its associated morphism is an isomorphism in C^{op}.
- The triple (z, π_x, π_y) is a product of x and y in C if and only if the associated triple is a coproduct of x and y in C^{op}.
- The object $*$ is a terminal object in C if and only it is an initial object in C^{op}.

We say that products are *dual* to coproducts and terminal objects are *dual* to initial objects.

Problem 2. In this problem, we will define the dual notion to a monomorphism. Say that a morphism $f : x \to y$ in C is an *epimorphism* if its associated morphism is a monomorphism in $\mathbb{C}^{\text{op}}$.

(a) Prove that a morphism $f : x \to y$ is an epimorphism if and only if, for any pair of morphisms $g, h : y \to z$ with $g \circ f = h \circ f$, we have $g = h$.

(b) Using this new definition, characterize the epimorphisms in the category of sets. (You've seen a property before which is equivalent to a function being an epimorphism.)

(c) Describe what an epimorphism is in as many categories as you can. (A very instructive but difficult example to consider is the category of groups.)

(d) Choose various categories of structured sets and structure-preserving maps, like vector spaces, rings, or topological spaces. Find an epimorphism in such a category whose underlying set-map is not an epimorphism. (Not all such categories will work.)

Problem 3. Prove that any morphism $* \to x$ is a monomorphism. Prove that any morphism $y \to \varnothing$ is an epimorphism.

Problem 4. Prove that there are isomorphisms

$$* \times x \cong x, \quad x \times * \cong x, \quad \varnothing + x \cong x, \quad \text{and} \quad x + \varnothing \cong x.$$

Problem 5. Suppose that the object 0 is both initial and terminal. Define a unique "zero arrow" $0_{x,y} : x \to y$ between any two objects, and show that the composition of zero arrows is a zero arrow. (Hint: there is such a 0 in the category of groups.)

Problem 6. Show that every isomorphism is both a monomorphism and an epimorphism. Then show via an example that a morphism may be both an epimorphism and a monomorphism, and yet not an isomorphism.

Problem 7. In this problem, we will study projection from a product.

(a) Show via an example that the projection maps $\pi_x : x \times y \to x$ are not always epimorphisms.

(b) By contrast, show that if there is a morphism $* \to y$, then π_x is an epimorphism.

Problem 8. Let $f : x \to y$ be a monomorphism. For any object z, define a map $f \times 1_z : x \times z \to y \times z$, and show that this is a monomorphism.

Problem 9. In this problem, we will study the distributivity of products over sums.

(a) Show that there always exists a morphism

$$(x \times y) + (x \times z) \to x \times (y + z);$$

we call this map the *distributor*.

(b) Find an example of a category in which the distributor is not always an isomorphism, and of a category in which the distributor is always an isomorphism. In the latter case, we say that category is *distributive*.

(c) Show that, if the distributor is an isomorphism, then there is a canonical isomorphism

$$x \times \varnothing \cong \varnothing.$$

Problem 10. State (and prove) the dual forms of Problems 7 to 9, i.e. the anglogous statements in the opposite category.